Original Title: Nekotoriy Issledsvaniya V Oblastiy Nelineiniy Kolebanii, Provedennyi V SSSR, Nachinaya S 1935
Published In: Uspekhi Fizicheskikh Nauk, Volume 33 Number 3, pp 335 – 352, 1947
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1947 Successes in the Physical Sciences Volume XXXIII Number 3

On The Thirtieth Anniversary of Soviet Physics

Some Research Results in the Field of the Nonlinear Oscillations, with the Lead Position Going to the USSR, Since 1935

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For the last ten years, the leading position in the field of nonlinear oscillations has essentially fluctuated[1] *). It appears that this will remain the present condition until such time as the research area expands further. Despite the pioneer works of B. van der Pol, E. Appleton and other contributors, this research arena remains poorly developed and little known; however, it is now possible to safely say, that the necessity of application of the nonlinear theory and nonlinear treatment for the diversity of oscillating problems has been successfully applied in various areas of modern technology, and has received wide recognition not only in scientific, but also in engineering circles. Alongside wireless and acoustics research, the theory of nonlinear oscillations has received the rights of citizenship in the labors of the electrical engineer, the aircraft technician, as well as in the technology of automatic control, which will be especially emphasized in this speech.

This expansion of the scope of the theory of nonlinear oscillations is the brightest feature of practical Soviet research over the past several years. Undoubtedly, it is a theoretical branch that is at our disposal and has become more perfect and effective in comparison with the initial efforts back in 1935, but it does not in essence contain any new ideas.

Let us briefly enumerate the basic elements of this topic.

1 – The qualitative (topological) theory of the differential equations created by H. Poincare[2], and the geometrical images provided by him (in phase space) of the various types of movements of dynamic systems, as, for example, the limit cycle representing established oscillations[3]. Research on auto-oscillations by means of this theory has led to new mathematical concepts of « structurally stable systems »[4].

2 – The theory of a series expansion based on a small parameter, developed in connection with problems of celestial mechanics (Euler, Lagrange, Poisson, Tisseran,

*) For a review of the research works completed up to 1935 see for example[1]

Liendshtedt, Poincare and so on), which allows one to calculate a periodic driving function, together with a method of slowly varying factors (or via van der Pol's method[5], which was the first to be applied to radio physical problems).

These quantitative methods applied in their most simple form are the most important for a case in radio physics of almost sinusoidal oscillations.

3 – The methods of research in regards to the stability of mechanically driven systems based on the works of Poincare and Lyapunov[6,7].

4 – The methods of reduction, in which the nonlinear associations entering into a problem (that consists of the performance of vacuum tubes, servomotors and so on) approximates a problem that is reduced by a series of rectilinear segments to "seaming", by which means that the certain conditions of the continuity, that includes in its solutions various systems of simple equations, are equal in various parts of its phase space.

This method has appeared especially effective for treatment of systems in which the mechanical driving cannot be considered approximately sinusoidal; such systems represent a special interest for the theory and applications of automatic control.

5 – About this last idea we will be discussing is not completely unimportant – it necessitates new physical language, and adequately describes properties of nonlinear systems that are absolutely distinct from the usual linear languages; this new nonlinear language was developed simultaneously when physicists seized upon the just enumerated mathematical methods and created, through their efforts, obviously new representations corresponding to their goals.

POORLY DEFINED NONLINEAR SYSTEMS

Let us remind readers first of all about some outcomes stated in the 1935 report. Emanating from the existence of a book entitled « the periodic solutions of the second type » by Poincare[6], with additional detailed discussions about phase, multiple phases, and the operating force, authored by L.I. Mandelstam and one of us (N.D. Papaleksi), we have come to the conclusion that the possibility of excitation and maintaining a certain working condition in the recycled system while being under an operation with harmonic EMF's, the multiple-phase oscillations are corresponding to a solution of the second type. The theory that these appearances have enveloped themselves in are not only characteristic of « a resonance of the second type », taking place in non-self-excited recycled systems, but also within many appearances observed in the self-excited system under one operational EMF or several harmonic EMF's.

Here, there are also additional concerns: synchronization on the overtone[7], an appearance of the suppression of oscillations,[8,9] auto-parametric or fractional resonances,[10] combinatory resonances[11,12] as well as an appearance of the so-called « asynchronous excitation » [9,13,14,15].

These same authors had for a lengthy period of time been developing the theory of parametrical generation of oscillations (by means of a periodic modification of parameters—capacity or self-induction) in systems with small depths of modulation and small nonlinearity indicators[16] and it also became clear there were further numerous examples with many additional characteristic appearances that were taking place via ion mode by the mechanical deriving of electrical energy.

By means of a combination specified above that highlighted quantitative and qualitative problem solving methods, including the concern for the behavior of regenerative systems both non-excited, and self-excited under an exterior operation with harmonic EMF's, these problems and methods have been investigated but there are issues that have remained that have not become satisfactorily resolved. An example of this in particular, regards a question at issue on the existence at synchronization on a threshold for an amplitude exterior to harmonic EMF's.

In all of the enumerated problems for the application of a method of perturbations or methods of small parameters as well as the van der Pol method, being applied in this supposition, one can say that in « a zero approximation » the considered systems are linear and conservative, i.e., that in their nonlinear terms, for these specific differential equations the more significant ones are the linear and not the conservative terms (damping) as well as the terms containing periodic parameters, when these are small enough.

Let's consider as an example a tank circuit with a variable capacitance varying under the law

$$\frac{1}{C} = \frac{1}{C_0}\left(1 + m\cos 2\omega t\right)$$

And the nonlinear resistance $R = R_0\left(1 + \gamma i^2\right)$, adjusted approximately on frequency with ω. We can write then the differential equation describing this system, in the form of

$$\frac{d^2q}{dt^2} + \omega^2 q = \left(\omega^2 - \omega_0^2\right)q - 2\delta_0\left[\frac{dq}{dt} + \gamma\left(\frac{dq}{dt}\right)^3\right] - m\omega^2\cos 2\omega t \cdot q \tag{1}$$

Or

$$\ddot{q} + q = -\mu\Delta q - 2\mu\vartheta_0\left(\dot{q} + \gamma\dot{q}^3\right) - \mu\cos 2\tau \cdot q \tag{2}$$

Where

$$\mu\Delta = \frac{\omega_0^2 - \omega^2}{\omega^2}; \quad 2\mu\vartheta_0 = \frac{R_0}{L\omega}; \quad \mu = m; \quad and \quad \tau = \omega t$$

We consider these magnitudes small – on the order of a smallness μ. According to Poincare[6], a required periodic solution of the equation (2) will differ little from one of the solutions of the equation

$$\ddot{q} + q = 0 \tag{3}$$

and its result will present false indications with regards to an aspect of some on degrees *m*, and the term with a zero index that would be one of the solutions of these equations (3) which we therefore name a « zero » as the initial solution.

Such a treatment has allowed us to discover a series of new system properties with periodically varying parameters, for example, with appearances that in parametrically connected systems where the research has led to the creation of a new type of motor with a slide control for the number of revolutions[17]; that concern the researchers here; also, the second parametrical resonance for a spectrum of frequencies 1:1 [18]; additionally, the appearances of parametrical combinational regeneration[19,20,21] which display the parametrical resonances that are taking place in connected systems[22].

However, for important practical cases, for example the parametrical generation of alternating currents, the developed theory appeared to be, in a quantitative sense, insufficient. The fact of the matter is that the magnitude of the potential, or EMF, of parametrical generators grows with the magnitude of the depth of modulation of the parameter and practically neither the depth of modulation of the parameter, or damping detuning $\omega^2 - \omega_0^2$ can be measured, as it is impossible to consider such a small number, and hence it is impossible to consider also Tomsonovskiy's initial system, i.e., to take the initial sinusoidal solution for the case of a system with one degree of freedom, or the sum of sinusoids for a system with many degrees of freedom.

It was condensed several years ago by L.I. Mandelstam, during the development with reference to these cases of reduced methodical forms of small parameters where it is quite naturally the initial approximations for the periodic solution of some linear differential equation with periodic factors.

Let's return to the case of the oscillating system considered above with periodically varying capacity. The equation system (1) can be presented in such an appearance

$$\ddot{q} + 2\vartheta_1\dot{q} + \frac{\omega_1^2}{\omega^2}(1 + m_1\cos 2\tau)q = \frac{\omega_1^2 - \omega_0^2}{\omega^2}q - 2(\vartheta - \vartheta_1)\dot{q} - 2\vartheta\gamma\dot{q}^3 + \left(\frac{\omega_1^2}{\omega^2}m_1 - m\right)\cos 2\tau \cdot q \tag{4}$$

If, on the one hand, one here chooses $\vartheta_1, m_1, and\, \frac{\omega_1^2}{\omega^2}$ so that the linear equation

$$\ddot{q} + 2\vartheta_1\dot{q} + \frac{\omega_1^2}{\omega^2}(1 + m_1\cos 2\tau)q = 0 \tag{5}$$

has a periodic solution, and on the other hand, $\frac{\omega_1^2 - \omega_0^2}{\omega^2} = -\mu\Delta, \quad 2(\vartheta - \vartheta_1) = \mu\vartheta_0 \;\; and \;\; m - m_1 = \mu$ as small and of the order of μ, and the magnitude γ, defining also the nonlinearity of the system, which is also small, then, following Poincare, it is possible to show, that the effort to approach the periodic solution of the equation (4) will lie near a straightforward periodic solution of the equation (5).

During L.I. Mandelstam's final months, he developed the theory of the approximate solution of a system of differential equations grounded on just this stated idea of periodic factors for any depths of modulation and small nonlinearity[23]. This theory was then applied to concrete cases of parametrically generating alternating currents.

Let us also mention the works of G.S. Gorelick[24] and S.M. Rytov[25] about non-stationary processes in systems with periodically varying parameters, where it has also spread to general applications in such systems where a method of slowly varying factors, such as the method of van der Pol, can be applied.

Along with those cases where parameters vary with phase, the comparisons are basically under an order of magnitude with the average characteristic phases of the system, in particular those cases when the phase of the modification of the parameters is very great and that has been deeply analyzed also. This special aspect of action gives rise to oscillations which are usually identified as modulated. Following closely the definition of modulated oscillation simply approximated as the oscillation slowly drifting away from the harmonic, S.M. Rytov has given some common treatments of both kinematic as well as dynamic modulation problems[26].

In this research it is possible to note two moments.

1. The method of perturbations is applied sequentially to problems of modulation. The small parameter μ is introduced thus as a factor at independent variable *t*. Modulated oscillation, is noted in the form of

$$s = A(\mu t)e^{i[\omega t + \varphi(\mu t)]}, \mu << \omega$$

Thus we have $\frac{dA}{dt}$ *and* $\frac{d\varphi}{dt}$ which are on the order of μ, i.e. A *and* φ are especially close to a constant rather than being less than μ. Various problems in regards to systems with modulated parameters have led to the equations whose factors vary from *t* through μt. For a solution of such problems the method of slow perturbations was developed, in which the zero approximation is similar to a quasi-stationary result that relates to an approximate conjugate solution. This last example represents a solution of the stationary case that is set with ($\mu = 0$), but additionally with the replacement of arbitrary constants of the slow functions *t*. This particular aspect of these functions is defined from the analysis of the subsequent approximations, and the modes serving for this purpose, are various, depending on the character of a problem. With reference to the nonlinear systems close to Tomasonovsky, the described procedure repeats, certainly, if one applies the van der Pol method, but for the linear modulated systems the aspect of a zero approximation is established differently (from the conditions of an orthogonality).

The method of measuring (slow but not necessarily small) perturbations essentially supplements the usual method of measuring (small and remaining arbitrary) perturbations.

2. There is a rather natural generalization given about the concept of modulation, i.e., the slow deviation from sinusoidality, relating to spatial and existential (wave) problems. As to a problem there that corresponds to the differential equations, quotients and derivatives, in which the factors or the boundary conditions that hold parameters that are dependent upon μx, μy, μz and μt. All of these outcomes are possible, certainly, and would be dependent upon the various orders of smallness on the one or several of these coordinates.

In relation to the wave equation, this statement of the question also envelops the usual passage of the approximation of geometrical optics (if the factor in most of the equations is spatially modulated, i.e., a velocity distribution), and with its diffraction impacting on enough smooth structures (if boundary conditions are modulated spatially). Application of the slow perturbations method to the Maxwell's equations (in the case of an inhomogeneous medium) has allowed us to obtain an approximation for the geometrical optics, alongside with the conservation law of a light beam, as well as the law of a modification of polarization along a ray, namely:

$$\frac{d\varphi}{ds} = \frac{1}{T}$$

where φ - is the angle between an electric vector and a principal normal to a ray, s – is a length of an arc measured along the ray, and T – is the radius of torsion of the ray.

Let us now pass on to another group of theoretical research concerning applications of the method of the small parameter.

Recently it has appeared possible to expand the circle problem, solved by this method, with the introduction of specific mathematical formulas to apply to the statement of some problems of not only « greater » magnitudes of the zero order ($\mu^0 = 1$) and « small » magnitudes of the positive order, but also « very much greater » magnitudes of negative orders ($1/\mu$, $1/\mu^2$, etc.). From a formal aspect it does not change the method for a small parameter, but from the physical point of view of the operation of « very much greater » magnitudes, it is not trivial and in some cases renders useful results.

The fact of the matter is that the physical reasons of greater frequency suggest to us that the interconnectedness of the relations is on the order of several magnitudes. It is certainly a question about which magnitudes to consider constant as $\mu \to 0$, i.e. to accept for magnitudes of zero order is a matter of consequence for any agreements. Each agreement of such kind is possible to have a certain equivalent choice regarding a specific considered aspect of the system and also have a zero approximation ($\mu = 0$). Therefore all similar samplings, being formally equivalent to each other, are not equivalent where discussing a physical model is concerned[27].

In this way it was possible to supply data points and resolve a problem about stabilization of frequencies of a vacuum tube generator by means of stabilizers (using a quartz oscillator, a tuning fork, as well as a volumetric resonator).

The vacuum tube generator connected with the quartz oscillator behaves essentially differently, than the generator with its two usual contours, both at the strong and at the weak connection. Thus already at most the frequency control in a method of small parameter it is necessary for a statement of the problem to reflect from the very beginning prominent features of quartz (a contour of the 2nd order), distinguishing it from the usual contour (1st order). Such a singularity of the stabilizer is of extremely great value $\sqrt{L_2/C_2}$. If one is to accept $\sqrt{L_2/C_2} \sim \mu^{-1}$, in view of $\sqrt{L_2/C_2} \sim 1$, it turns out: $L_2 \sim \mu^{-1}$, $C_2 \sim \mu$, i.e. the inductance of quartz appears to have a « very big » magnitude. Owing to the equations for currents I_1 and I_2 (accordingly in a contour of the generator (and, in an equivalent contour for quartz) becomes*)

$$\left.\begin{aligned} \ddot{I}_1 + I_1 &= -\mu\vartheta_1\dot{I}_1 - \mu x_1\ddot{I}_2 + \mu I_a - \mu\Delta I_1 \\ \ddot{I}_2 + I_2 &= -\mu^2\vartheta_2\dot{I}_1 - \mu^2 x_2\ddot{I}_1 \end{aligned}\right\} \tag{6}$$

The prominent feature exhibited by equation (6) is caused by the introduction of $L_2 = 1/\mu$, where this is an asymmetry of orders of smallness not only by decrements, but also via connection factors. This circumstance enables the possibility to construct by means of a small parameter method, the strict nonlinear theory of frequency control**). Besides, this turns out to be the somewhat obvious picture of a stabilization process. Namely, if we consider the terms of the equations in (6) which are not containing μ, (thus having a zero approximation) we receive two independent conservative linear oscillators, both with identical frequency. At the accounting of terms with μ (to the first approximation) we have the self excited generator, which is almost on a resonance (detuning $\mu\Delta$) operates with a driving force - $\mu\chi_1\ddot{I}_2$, i.e. this turns out to be a problem about coherence. A radiant

*) Here too we introduce dimensionless time $\tau = \omega t$, where $\omega = 1/\sqrt{L_2C_2}$ is the quartz frequency. Decrements $\mu\vartheta_1, \mu^2\vartheta_2$ and the factors of connection $\mu x_1 = \mu/L_1$, $\mu x_2 = M/L_2$ have different orders, as do resistances R_1, R_2 where the order μ for a coefficient for mutual induction is $M \sim \mu$, ($L_1 \sim 1$, $L_2 \sim 1/\mu$).

**)This theory is developed and quantitatively confirmed from experimental experience for the basic schemes of stabilization (tightenings and oscillators) in[28]. Except for that in[29], the small parameter method it is developed in a general view for systems with two degrees of freedom and with those equations that contain terms of various orders of smallness.

with this force impacts the second oscillator as before even if there are independent and conservative factors indicated. At last, to account for the terms of the second order, simultaneously, we would enter some damping into the quartz, as well as supporting oscillations in the quartz through action from the generator. Thus this interaction is asymmetric: the generator acts on the quartz very poorly ($\sim \mu^2$) according to a small decrement of quartz, and quartz acts on the generator equally as poorly ($\sim\mu$) and consequently seizes (Editor's note: phase locks) when near a resonance level point.

The precise language sense of small parameter gains is also a concept that is conditional with frequency control, namely: it is in such a self-oscillatory condition at which, as a result de-tunings of a contour by way of μ, the frequency deviates from a constant value not more strongly, than by way of μ^2.

The expansion of the method of small parameters briefly described here has appeared useful, but not only with problems about stabilization of frequencies. As an example, it is possible to specify the alternating current theory of generators where the assumption of « very big » inductance stators also allows us to finish all the calculations, thus preserving all of the features of interest of this considered example.

The method of small parameters also has been applied to a solution of some problems, especially those concerning distributed parameter systems (nonlinear problems in partial derivatives) which have gained special value in connection with such successes, reached in the field of technology in regards to very high frequencies[30,31,32].

STRONG NONLINEAR SYSTEMS

One of the basic tendencies of the research developments discussed here consists of, having begun with problems of radio-physics, those achievements and application protocols that were spread to an area of endeavor which at first sight seems rather far from it like the field theory of automatic control. This theory represents a wide field for the application and developments of physical ideas as well as the mathematical methods – which have become usual for radio-physics – to be engaged in by research into auto-oscillations. In spite of the great value which has been gained with efforts on relaxation oscillations, the radio-physicist, for abundantly clear reasons, saw infinitely more in its purpose: he is interested in almost sinusoidal oscillations generated by poorly defined nonlinear systems. As already mentioned above, the theory of automatic control mostly deals with strongly nonlinear systems in which auto-oscillations, if they exist, are as essential as they are non-sinusoidal.

Let us remind ourselves all over again of the simple example taken from the area of radio-physics[33], a method of « seaming » the trajectories in a phase space, about which we had already made mention of in the introduction. After that we can briefly consider some final works that are not concerned with linear problems for automatic regulating.

To avoid distorting the historical perspective, the following is a necessary criticism beforehand. Ten years prior to the origin of wireless, the French engineer Leaute[34], was studying auto-oscillations in some device of automatic control, and actually investigated the phase space of this device from which he had traced its integral curves and limit cycles (but not having given it this labeling: although it was extant, Leaute was not familiar with the work of Poincare, who, a little bit earlier, had published about limit cycles for the first time it ever appeared in mathematics). For reasons about which we here will not speak, the remarkable works of Leaute have almost been completely forgotten. Research about this topic will be brief in this speech, representing itself a new application of methods used in radio-physics, which are at the same time a demonstration of the persistence of the works of Leaute.

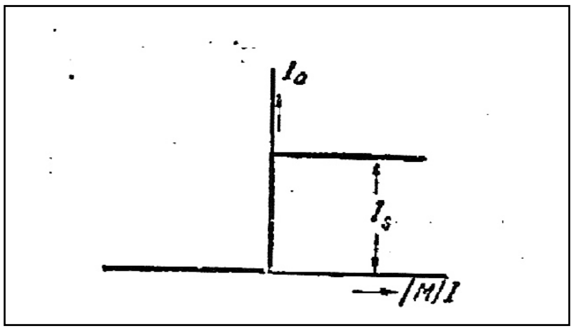

Figure 1

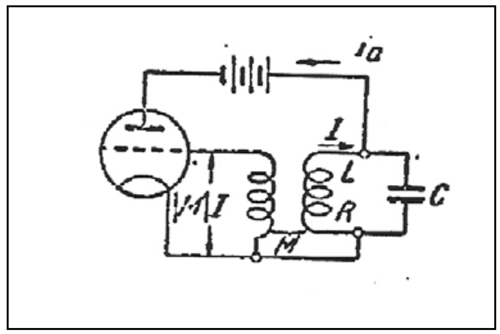

Figure 2

If we idealize the performance of a vacuum tube such that it is specified as shown in Fig.1, the differential equation of the system whose schematic is shown in Fig. 2, will look like:

$$L\ddot{I} + R\dot{I} + I/C = \begin{cases} I_s/C, & when \quad \dot{I} > 0 \\ 0, & when \quad \dot{I} < 0 \end{cases} \tag{7}$$

The phase space represents a plane $I, \dot{I}$. The half plane $\dot{I} > 0$ is filled by segments of the trajectories representing oscillations, damping around a position of equilibrium $I = I_s, \dot{I} = 0$; the half plane $\dot{I} < 0$ contains segments of the trajectories corresponding to the oscillations damping around the position of an equilibrium $I = 0, \dot{I} = 0$. Segments of the trajectories should be incorporated on an axis $I = 0$ so that $I \ and \ \dot{I}$ remain continuous.

Take I_n so there will be an *n*-th abscissa intersection of a trajectory with a ray $D(\dot{I} = 0, I > 0)$. An abscissa (*n*+1)-th that intersects with that ray will be

$$I_{n+1} = I_n e^{-kT} + I_s\left(e^{-\frac{kT}{2}} + 1\right); \quad k = \frac{R}{2L}, T = \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} \tag{8}$$

The equation (8) can be interpreted as a mathematical expression of the discussion of the Leaute method (Fig 3). This equation expresses the point-wise transformation of a straight line. The thick straight line corresponds to the equation (8) and the thin one makes an angle of 45 degrees with the axes of the coordinates. It is easy to see that any I_0 *and* $n \to \infty$ of any of the I_n series that aspire to a final limit will have the form:

$$I^* = \frac{I_s}{1-e^{-kT/2}}$$

It is those « amplitudes » of the limit cycle that represent the established oscillations on a phase plane *).

The auto-oscillations arising in many automatic control devices can be studied similarly. Let us consider an example from the work of Andronov, Bautin and Gorelick[35], concerning, though strongly simplified, the quite modern problem although with a few differing aspects that echoes the concepts described in Leaute's problem. Atmospheric matter (Editor's note: air) passes through the airscrew, which has an automatically controlled variable pitch. The servomotor rotates with an angular velocity ω according to the equation

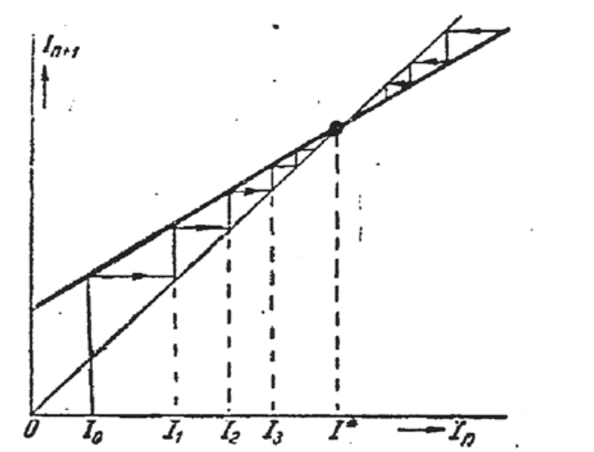

Figure 3

$$I\dot{\omega} = P(\omega,\lambda) - Q(\omega,\varphi); \qquad (9)$$

Here *I* – is a moment of inertia, and *P* – is a driving moment, *Q* – is a moment of resistance, λ – is a parameter describing the feeding of a gas, φ - is the angle of the turn of the airscrew's blades while the servomotor operates via a centrifugal tachometer, this all results in that the airscrew's pitch changes under the law

$$\dot{\varphi} = F(\xi) \qquad (10)$$

Where ξ - is the displacement clutches of the tachometer. An ideal would be to make ξ a single valued function about ω, and *F*(ξ) such that it was equaled to zero only at $\omega = \omega_n$ where ω_n – is a demanded value of an angular velocity which should remain constant. In that case we had

*) Periodic solutions of the equation $L\ddot{I} + I/C = 0$ are very delicate formations: they disappear as soon as we introduce, for example a captive $R\dot{I}$ as though we have a very small *R*. On the contrary, the periodic solution (7) continues to exist with not much greater modifications of the differential equation. The concept a structurally stable system (see introduction) generalizes this property which can form the foundation for the mathematical definition of auto-oscillations.

at a stationary condition $(\dot{\omega} = 0, \dot{\varphi} = 0)$ by virtue of equation (10) the function $F(\xi) = 0$ or $\omega = \omega_n$ by virtue of equation (9) where φ consequently and by force of draft becomes a function of λ. The corresponding choice of the parameter's values guarantees in this case a stability in regards to this stationary condition.

This ideal is never carried out however. First, the tachometer possesses inertia and friction. Second, the servomotor possesses a dead zone. Function $F(\xi)$ can have, for example, an aspect as specified in Figure 4. These circumstances can change the properties of the system completely. Its behavior has undergone extensive research in considered work at the following supposition: it is possible to neglect the inertia of, but not the friction of, a tachometer.

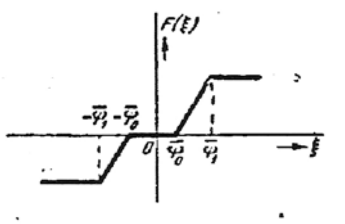

Figure 4

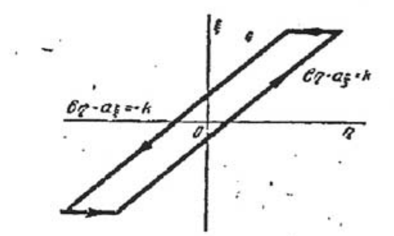

Figure 5

Let's describe first of all the dynamics of a tachometer, including friction which follows Coulomb's law. As under the supposition, the tachometer has no inertia, although there is an equilibrium between the force of friction R and the equally effective centrifugal and restoring forces:

$$R + F(\xi) = \begin{cases} F = -k, & if \quad \dot{\xi} > 0 \\ 0 < |F| < k, & if \quad \dot{\xi} = 0 \ (k > 0), \\ F = +k, & if \quad \dot{\xi} < 0 \end{cases}$$

Assuming, that the factors being considered in this driving function are so small, that R can be a linear function $(R = b\eta - a\xi;\ \eta = \omega - \omega_N;\ a > 0, b > 0)$ as we can see below:

$$\begin{aligned} & b\eta - a\xi = +k, \quad \dot{\xi} = \dot{\eta} \quad if \quad \dot{\xi} > 0 \\ & |b\eta - a\xi| < k, \quad if \quad \dot{\xi} = 0 \\ & b\eta - a\xi = -k, \quad \dot{\xi} = \dot{\eta} \quad if \quad \dot{\xi} < 0 \end{aligned} \tag{11}$$

When the angular velocity makes the oscillation, the representing point on plane ξ, η the cycle that results is shown in Figure 5. With the « shift in phase » between ξ and η, it is possible to consider this as a physical reason for the origin of the auto-oscillations.

Linearizing (9), we obtain:

$$\dot{\eta} = -M\eta - N\varphi \qquad (M > 0, \quad N > 0)$$

The equation (10) can be, according to Figure 4, noted in the form of

$$F(\xi) = -F(-\xi) = \begin{cases} 0, & if \quad |\xi| < \overline{\psi}_0 \\ \overline{a}(\xi - \overline{\psi}_0), & if \quad \overline{\psi}_0 < \xi < \overline{\psi}_1 \\ +A, & if \quad \xi > \overline{\psi}_1 \end{cases}$$

Let us suppose:

$$\frac{a\xi}{2k} = x, \qquad \frac{b\eta}{2k} = y, \qquad \frac{-b}{2k}\left(\eta + \frac{N}{M}\right) = z, \quad Mt = \tau,$$

$$\frac{a\overline{\psi}_0}{2k} = \overline{\psi}_0, \quad \frac{a\overline{\psi}_1}{2k} = \overline{\psi}_1, \quad \frac{Nb}{2kM^2}\overline{A} = A, \qquad \left(\alpha = \frac{A}{\overline{\psi}_1 - \overline{\psi}_0}\right)$$

We can now write the differential equations

$$\frac{dx}{d\tau} = f(x, y, z); \quad \frac{dy}{d\tau} = z; \quad \frac{dz}{d\tau} + z = g(x) \tag{I}$$

Where functions *f* and *g* are also certain as follows in a three-dimensional « degenerate » phase space organized by half-planes $H\left(x - y = \frac{1}{2}, z < 0\right)$ and $H'\left(x - y = \frac{1}{2}, z > 0\right)$ and a stratum $G\left(|x - y| < \frac{1}{2}\right)$:

$$f(x, y, z) = \begin{cases} z & in \quad H \ and \ H' \\ 0 & in \quad G \end{cases} \tag{II}$$

$$g(x) = -g(-x) = \begin{cases} 0, & if \quad |x| < \psi_0 \\ \alpha(x - \psi_0), & if \quad \psi_0 < x < \psi_1 \\ A, & if \quad x > \psi_1 \end{cases}$$

By virtue of the equation series in (II) the nonlinear system as described in (I) has led to linear versions via the previously described equations, which are real in the various areas of phase space. System (I) and the condition of the x, y, z, coordinates on the continuity boundaries of these areas completely define the driving function via the represented points.

In Figure 6 a thin line segment of a typical trajectory is shown. It looks like a spiral; we have illustrated here a driving oscillation. While the tachometer is displaced, the representing point is on *H* or *H'*; at the stopping of the servomotor it moves on a straight line when the servomotor is in operation on a curve. When angular acceleration changes sign, the tachometer – because of friction – for an instant stops, the representing point interferes in a stratum *G* and describes an arc in a plane x=const. Then the tachometer again comes into action, and the representing point again moves on *H'* or *H*.

It is necessary to know, whether the spiral is displaced or torn and whether there are also limit cycles. This problem is solved by means of the calculations, which have similar approaches to ones that have been made by us for the vacuum tube generator. Through this means the phase space – becomes three-dimensional, but it is so deteriorated that all trajectories intersect properly with the chosen ray plane *H* or *H'*, for example ray $L\left(x - y = \frac{1}{2}, z = 0, x < -\psi_0\right)$. Thus, it is enough to investigate point-wise the transformation of this ray to its utmost, certainly (I) and (II) and conditions of the « seaming » of trajectories being satisfied.

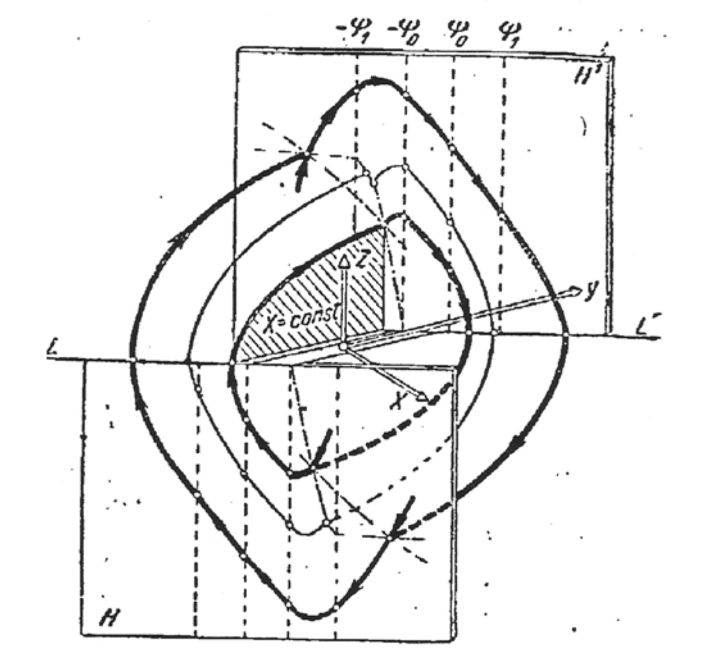

Figure 6

The calculations show that the oscillations damp out at any entry conditions, or as in (Figure 6) there are two limit cycles – one steady and the second, of smaller size that is unstable (they are shown by the thick lines). In the second case it would damp out only weak enough perturbations. In Figure 7 it is shown in the parameter space (α, ψ_0, A) in which such auto-oscillations are possible. It is concluded between a plane of $\psi_0 = 0$ and a cylindrical surface [*A*], forming which axes are parallel to *A* and a surface [α] containing an axis *A*. It is always possible to save auto-oscillations or a diminution of *k* or *N* or the magnification of a dead zone.

Andronov, Bautin, and Gorelick's other work,[36] in the main, highlighted the problem concerning the large class of devices relating to automatic control and the general understanding of the case just considered. Of the key points to know about these systems, here are a few to note: to which the velocity of the servomotor copes not only with the device measuring the governed magnitude (for example a tachometer), but also, as presented for the first time by the French engineer Farko, a linear combination of both the displacement of the measuring device and the servomotor that allows the equipment to more effectively contend with occasions of instability. Two cases for which it has appeared possible to plot inside these boundary space parameters between areas of stability of a stationary condition, on the one hand, and areas in which it is possible to have auto-oscillations with another, have undergone further research: a case where the servomotor has linear performance and a case of the servomotor possessing a constant velocity via an absolute value and a dead zone.

Till now, it was a question of problems in which a phase space had so simple an appearance, though also three-dimensional, that its research was reduced to a point-wise transformation of a line and to a line. However, the large number of very important problems leads to « full » three-dimensional phase spaces and that to understand during integral curves, it is necessary to subject

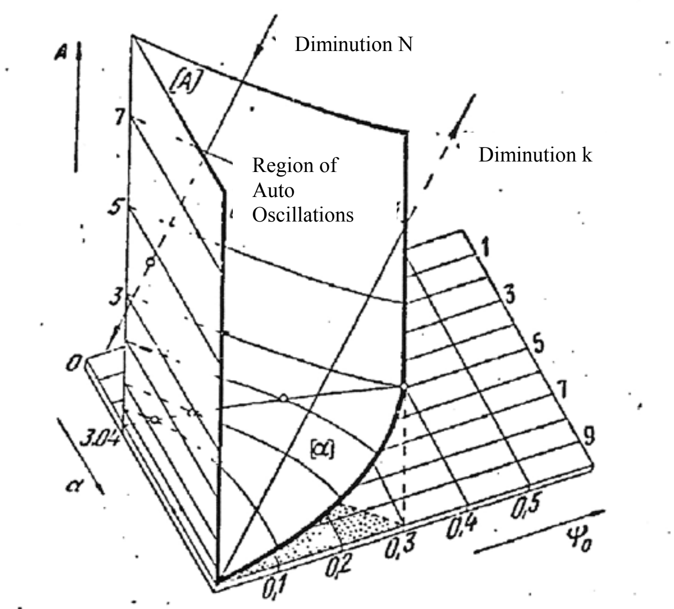

Figure 7

to research the point-wise surface to surface transformation. Wischnegradski's classical problem, which together with Maxwell is the founder of the dynamic theory or automatic control, is an example. Here it is necessary to make still one further short historical detour.

In 1841, the well-known astronomer Airy had studied auto-oscillations, which the clockwork intended for maintaining a uniform equatorial motion in a telescope had been his subject of interest. He could not however provide a satisfactory theory for this application. Maxwell, in the well-known memoir « On Governors », published in 1868,[37] investigated radicals of a secular equation linearizing the equations of motion for systems of direct regulation *) for the first time

*) That is, without the servomotor: the measuring device operates immediately off of the adjusting mechanism.

having formulated what we now call the oscillation conditions of self-excitation. There exists an immediate continuity between the research efforts of Maxwell about the conditions of self-excitation of auto-oscillations and the later, widely known researches of Routh[38] about the stability of driving functions to achieve such oscillations.

Wischnegradski was interested in the problems of regulating oscillations not only from the vantage point of the scientist, but also that of the engineer. Wischnegradski, unlike Maxwell, (who was not familiar with Wischnegradski's memoir[39]), finally, in 1886, described the problem of direct regulating in that aspect, a technique in which he was singularly interested in at the time. Wischnegradski's problem remains, even to this day, one of the primary problems of regulatory theory.

Let's consider the machine, for example one driven by steam, in which the moving moment *P* depends on position *y*. A governing mechanism and the moment of resistance *Q* is a constant:

$$\dot{I}\omega = P(y) - Q \tag{12}$$

The governing mechanism driven by a tachometer, where inertia and friction cannot be neglected:

$$\begin{gathered} y = \alpha x, \\ m\ddot{x} + kx + f(\dot{x}) = \beta(\omega - \omega_N) \end{gathered} \tag{13}$$

Friction $f(\dot{x})$ develops from viscous friction and general friction following Coloumb's law. The stationary condition $(\dot{\omega} = 0, \dot{x} = 0)$ can be steady or unstable. It is unstable, in particular, when oscillations are caused by inertia and the elasticity of the tachometer, increase contrary to friction owing to interaction between the tachometer and the machine. It is required to discover the conditions of stability in connection with these pieces of equipment. Wischnegradski's problem is succinctly described and presented by these equations. This problem is definitely nonlinear- in both description and solution.

Wischnegradski in his well known memoir has himself given a solution for the linear case, when $f(\dot{x}) = \gamma\dot{x}$. For a special case, when viscous friction is absent, a series of outcomes has been described by Lecornu[40], Zhukovskiy[41], and Mises[42]. We name this case the Mises' problem. As to Wischnegradski's problem, it remains unsolved.

This subsequently led to the examination of the three-dimensional phase space with special emphasis on the examination of dot transformation of a plane in a plane, where Andronov and Maier had a new opportunity to provide the solutions to Mises' problems[43], but also have provided a pathway mathematically to completely solve Wischnegradski's problem as well[44].

The parameter space which at an appropriate selective point can be led to be two dimensional, and broken into three fields whose boundaries were possible to be calculated: 1) a field in which the system spires to be stationary as to its condition, it can be at any initial condition; 2) a field, in which the system is unlimited leaves from a stationary condition at any starting condition; 3) a field, depending on the staring conditions, in which the system comes nearer to a stationary condition, or from it leaves. It has appeared also possible to calculate for each value of parameters belonging to this peak field perturbation, with which else the automatic control device can consult.

Let's specify one more three-dimensional nonlinear problem investigated via the method of transformation of a plane in a plane; the problem about stabilization on a course by an aircraft via auto-pilot[45].

Let the angles describing the position of the aircraft and its rudder, be φ and η then we can write the equations in the form of

$$\ddot{\varphi} + M\varphi = -N\eta; \qquad \dot{\eta} = F(\psi)$$

Where M and N –positive constants and $F(\psi)$–the characteristic of the servomotor. It is supposed, that this last function is operated in a linear combination:

$$\psi = \varphi - \alpha\eta + \beta\dot{\varphi}$$

where α, β – are positive constants. The servomotor differs in its constancy of velocity and the dead band region. The solution of this problem has shown that the parameter space is divided into a field in which there are auto-oscillations of the aircraft around its course, with the second field having feeble auto-oscillations and with the third area with strong auto-oscillations.

Let's give still other works concerning problems of automatic control[46,47]

Research of the iterated transformations of the aspect

$$x_n = f(x_n - 1) \qquad (n = 1,2,3....),$$

which is indeed playing a much greater role in this part of our review, has allowed us to investigate other problems about nonlinear oscillations[48,49,50,51,52], and in particular, about those auto-oscillations of systems with distributed constants such as for example a violin string driven by a bow, or the Laherovskiy system driven by a vacuum tube[51]. (The problem is reduced to research of sequential transformations of values of some function, satisfying the wave equation, and this transformation is set by some nonlinear boundary conditions). To a similar given problem where Bovscherverov systems[53] were investigated with the so-called « late feedback » of which these examples can serve: the echo-location of Jaques Bodin[54] and the Koulikoff-Chilowsk scheme for the measurement of distances by means of radiowaves[55]. Recently a similar method was developed by Andronov and Gorelick that had considered a nonlinear problem about a resonance relating to a relativistic particle, moving inside of a cyclotron[56].

Some Researches in the Field of Nonlinear Oscillations

Here the nonlinearity is caused by the dependence of the particle's mass upon the (Editor's note: relativistic) velocity. Undoubtedly, the methods of applying the theory of nonlinear oscillations, are fated to find a wide application in the examination of the movements of electrons and ions in particle accelerators, as well as playing such an important role in the up-to-date techniques of physical experiments, such as the electron's motion in devices intended for ultra-high frequency oscillations (Editor's note: magnetrons).

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